GENUS FORMULAS FOR ABELIAN $p$–EXTENSIONS

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ABSTRACT. We apply a result of E. Kani relating genera and Hasse–Witt invariants of Galois extensions to a family of abelian $p$–extensions. Our formulas generalize the case of elementary abelian $p$–extensions found by García and Stichtenoth.

1. Introduction. E. Kani proved in [2] that if $L/K$ is a finite Galois extension of function fields with Galois group $G$, then any relation among idempotents of subgroups of $G$ in $\mathbb{Q}[G]$ implies the same relation among the quotient genera. The quotient genus for a subgroup $H$ of $G$ is the genus of the field $K_H := L^H$.

In the same paper, Kani proved that if the field of constants $k$ of $K$ is a field of positive characteristic $p > 0$, then any relation among the subgroups $H$ of $G$ implies the same relation among the Hasse–Witt invariants of the fields $K_H$.

In this paper we consider an arbitrary field $k$ of characteristic $p > 0$, a function field $K$ with field of constants $k$ and a Galois extension $L/K$ with Galois group isomorphic to $(\mathbb{Z}/p^m\mathbb{Z})^n$ where $m$ and $n$ are natural numbers. We find two formulas relating the genus $g_L$ of $L$ and the genera of a family of subextensions. The first one, is the family of all cyclic subextensions of $K$ and the second, the family of all subextensions $E$ with $L/E$ cyclic. The same relations hold for the Hasse–Witt invariants. Our results generalize the formula found by García and Stichtenoth [1] for elementary abelian $p$–extensions.

2. The results. Let $k$ be any field of positive characteristic $p$ and let $K$ be a function field with field of constants $k$. Let $L/K$ be a Galois

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extension with Galois group isomorphic to $G = (\mathbb{Z}/p^m\mathbb{Z})^n$. Let $\mathcal{G}$ be the set of all subgroups of $G$. For each $H \in \mathcal{G}$, let $K_H$ be the subfield of $L$ fixed by $H$, that is $K_H := L^H$. Let $g_H$ be the genus of $K_H$ and let $\tau_H$ be the Hasse–Witt invariant of $K_H$. For $H \in \mathcal{G}$, let $\epsilon_H$ be the norm idempotent of $H$:

$$\epsilon_H := \frac{1}{|H|} \sum_{h \in H} h \in \mathbb{Q}[G].$$

In [2], E. Kani proved the following result.

**Theorem 2.1** (E. Kani). Any relation

$$\sum_{H \in \mathcal{G}} r_H \epsilon_H = 0 \quad \text{with} \quad r_H \in \mathbb{Q},$$

among the norm idempotents yields the following two relations

$$\sum_{H \in \mathcal{G}} r_H g_H = 0 \quad \text{and} \quad \sum_{H \in \mathcal{G}} r_H \tau_H = 0,$$

among the genera and among the Hasse–Witt invariants. □

Let $\mathcal{H}_i$ be the set of all subgroups of $G$ isomorphic to $(\mathbb{Z}/p^m\mathbb{Z})^{n-1} \oplus (\mathbb{Z}/p^{m-i}\mathbb{Z})$, $0 \leq i \leq m$. The set of the fields fixed by $H \in \mathcal{H}_i$ is the set $\mathcal{K}_i$ of all the subfields $K \subseteq E \subseteq L$ such that $\text{Gal}(E/K) \cong (\mathbb{Z}/p^i\mathbb{Z})$, that is, the collection of all the cyclic extensions of $K$ of degree $p^i$ contained in $L$. Our main result is

**Theorem 2.2.** We have the following relations

$$g_L = -p\left(\frac{p^{n-1} - 1}{p - 1}\right) g_K - (p^{n-1} - 1) \sum_{i=1}^{m-1} \sum_{E \in \mathcal{K}_i} g_E + \sum_{E \in \mathcal{K}_m} g_E,$$

and

$$\tau_L = -p\left(\frac{p^{n-1} - 1}{p - 1}\right) \tau_K - (p^{n-1} - 1) \sum_{i=1}^{m-1} \sum_{E \in \mathcal{K}_i} \tau_E + \sum_{E \in \mathcal{K}_m} \tau_E.$$
Corollary 2.3 (Garcia–Stichtenoth [1]). If $L/K$ is an elementary abelian \( p \)-extension of degree \( p^n \), we have

\[
g_L = -p \left( \frac{p^{n-1} - 1}{p - 1} \right) g_K + \sum_{E \in \mathcal{K}_i} g_E. \tag{2.5}
\]

Now let \( T_i \) be the set of cyclic subgroups of \( G \) of order \( p^i \), \( 0 \leq i \leq m \). Let \( \mathcal{L}_i \) be the set of subextensions \( K \subseteq E \subseteq L \) such that \( L/E \) is a cyclic extension of degree \( p^i \). We have \( \mathcal{L}_i = \{ E \mid E = L^H \text{ with } H \in T_i \} \).

Then

\[\text{Theorem 2.4. We have the following relations}
\]

\[
p \left( \frac{p^{n-1} - 1}{p - 1} \right) g_L = -p^m g_K - \left( p^{n-1} - 1 \right) \sum_{i=1}^{m-1} p^i \sum_{E \in \mathcal{L}_i} g_E + p^m \sum_{E \in \mathcal{L}_m} g_E,
\]

and

\[
p \left( \frac{p^{n-1} - 1}{p - 1} \right) \tau_L = -p^m \tau_K - \left( p^{n-1} - 1 \right) \sum_{i=1}^{m-1} p^i \sum_{E \in \mathcal{L}_i} \tau_E + p^m \sum_{E \in \mathcal{L}_m} \tau_E.
\]

Remark 2.5. The genera of the subfields considered in Theorem 2.2 can be computed using the results of H. L. Schmid [3].

It is not easy to use Theorem 2.4 in applications since the family of fields considered is in the top of the extension, so the genera is hard to find.

3. The proofs. First we consider

\[(3.1) \quad M_i := \sum_{H \in \mathcal{H}_i} \epsilon_H, \quad 0 \leq i \leq m. \]

Note that \( M_0 = \sum_{H \in \mathcal{H}_0} \epsilon_H = \epsilon_G = \frac{1}{p^m} \sum_{\sigma \in G} \sigma \).

Fix an element \( \sigma \in G \). Let \( T(i, \sigma) \) be the number of distinct subgroups \( H \in \mathcal{H}_i \) such that \( \sigma \in H \). That is,

\[T(i, \sigma) := |\{ H \in \mathcal{H}_i \mid \sigma \in H \}|.\]
Let $s$ be a natural number $1 \leq s \leq m$ and let

$$G_s := \{ \sigma \in G \mid o(\sigma) = p^s \}.$$ 

Note that given any element $\sigma \in G_s$, there exists an element $\tau \in G$ of order $p^m$ such that $\tau^{p^{m-s}} = \sigma$. If $\theta$ and $\sigma$ are two elements of $G_s$, then there exists an automorphism $\Phi \in \text{Aut}(G)$ such that $\Phi(\theta) = \sigma$. Thus $T(i, \sigma) = T(i, \theta)$. Therefore, it makes sense to define

$$(3.2) \quad T(i, s) := T(i, \sigma),$$

where $\sigma$ is any element of $G_s$.

Let $C_s := \sum_{\substack{\sigma \in G_s \sigma \in \mathbb{Q}[G]}}$. Then

$$M_i = \sum_{H \in \mathcal{H}_i} \frac{1}{|H|} \sum_{h \in H} h$$

$$= \frac{1}{p^{m(n-1)+(m-1)}} \sum_{s=0}^{m} T(i, s) \sum_{\sigma \in G_s} \sigma$$

$$= \frac{1}{p^{nm-i}} \sum_{s=0}^{m} T(i, s)C_s.$$

We need to compute $T(i, s)$ for all $0 \leq i, s \leq m$. To this end, let $e_s$ be the number of elements of $G$ of order $p^s$. We have

$$e_s = q^s - q^{s-1}, \quad 1 \leq s \leq m, \quad \text{and} \quad e_0 = 1,$$

where $q = p^n$. In particular if $h_i$ is the number of distinct cyclic subgroups of $G$ of order $p^i$, it follows that

$$h_i = q^i - q^{i-1}, \quad 1 \leq i \leq m, \quad \text{and} \quad h_0 = 1.$$

Since in an abelian group its lattice of subgroups is symmetric, that is, if $B$ is a subgroup of a finite abelian group $A$, then $A$ contains a subgroup isomorphic to $A/B$, it follows that

$$h_i = |\mathcal{H}_i|.$$ 

Let $H \in \mathcal{H}_i$ and let $L(H, s) = |H \cap G_s|$. Since all subgroups in the collection $\mathcal{H}_i$ are isomorphic, it makes sense to define

$$L(i, s) := L(H, s),$$
where $H$ is any subgroup in $\mathcal{H}_i$.

Let $F \subseteq \mathcal{H}_i \times G_s$ be defined by

$$F := \{(H, \sigma) \mid \sigma \in H\}.$$ 

We can compute $|F|$ either column by column or row by row which gives us:

$$|F| = h_i L(i, s) = T(i, s)e_s,$$

respectively. That is, to find $T(i, s)$ it suffices to find $L(i, s)$.

Now fix $H \in \mathcal{H}_i$ and let $B_s := \{x \in H \mid x^{p^s} = \text{Id}_G\} = \{x \in H \mid o(x) \text{ divides } p^s\}$. Then $L(i, s) = |B_s| - |B_{s-1}|$ for $1 \leq s \leq m$ and $L(i, 0) = |B_0| = 1$. Now to find $B_s$, note that $B_s = \ker \Psi$, where $\Psi : H \to H, \Psi(x) = x^{p^s}$. The image of $\Psi$ is $H^{p^s}$. Hence

$$|B_s| = \frac{|H|}{|H^{p^s}|}.$$ 

Since $H \cong (\mathbb{Z}/p^m\mathbb{Z})^{n-1} \oplus (\mathbb{Z}/p^{m-s}\mathbb{Z})^{n-1} \oplus A$, we have $H^{p^s} \cong (\mathbb{Z}/p^{m-s}\mathbb{Z})^{n-1} \oplus A$, where

$$A \cong \begin{cases} (\mathbb{Z}/p^{m-i-s}\mathbb{Z}) & \text{if } 1 \leq s \leq m - i, \\ 0 & \text{if } m - i < s \leq m. \end{cases}$$

Therefore we have

$$L(i, s) = \begin{cases} 1 & \text{if } s = 0, \ 0 \leq i \leq m, \\ p^{n(s-1)}(p^n - 1) & \text{if } 1 \leq s \leq m - i \ (0 \leq i \leq m - 1), \\ p^{(n-1)(s-1)+(m-i)}(p^{n-1} - 1) & \text{if } m - i + 1 \leq s \leq m \ (1 \leq i \leq m). \end{cases}$$

From (3.3) and (3.4), we obtain

$$T(i, s) = \begin{cases} 1 & \text{if } i = 0, \ 0 \leq s \leq m, \\ h_i & \text{if } s = 0, \ 0 \leq i \leq m, \\ \frac{p^{n-1} - 1}{p^{n-1}}p^{(n-1)(i-1)} & \text{if } 1 \leq s \leq m - i, \ (1 \leq i \leq m - 1), \\ \frac{p^{n-1} - 1}{p^{n-1}}p^{(n-2)(i-1)+(m-s)} & \text{if } m - i + 1 \leq s \leq m \ (1 \leq i \leq m). \end{cases}$$
Thus, from (3.5), we obtain

\[
M_i = \frac{p^i}{pnm} h_i \text{Id}_G + \frac{p^i}{pnm} \sum_{s=1}^{m-i} \left( \frac{p^n - 1}{p - 1} \right) p^{(n-1)(i-1)} C_s
\]

\[
+ \frac{p^i}{pnm} \sum_{s=m-i+1}^{m} \left( \frac{p^n - 1}{p - 1} \right) p^{(n-2)(i-1)+(m-s)} C_s,
\]

for \(1 \leq i \leq m\) and \(M_0 = \epsilon_G\).

Now, in order to obtain a relation among the norm idempotents, since \(M_0 = \epsilon_G\) and \(\text{Id}_G = \epsilon_\text{Id}_G\), what we need is to find \(x_1, \ldots, x_m \in \mathbb{Q}\) such that

\[
\sum_{i=1}^{m} x_i M_i = y_0 \text{Id}_G + \sum_{s=1}^{m} y_s C_s,
\]

with \(y_0 \in \mathbb{Q}\) and \(y_1 = y_2 = \cdots = y_m \neq 0\).

Let \(x_1, \ldots, x_m \in \mathbb{Q}\) and

\[
\sum_{i=1}^{m} x_i M_i = \left( \sum_{i=1}^{m} \frac{p^i}{pnm} x_i h_i \right) \text{Id}_G + \left( \frac{p^n - 1}{p - 1} \right) \sum_{i=1}^{m-1} \sum_{s=1}^{m-i} x_i \frac{p^{(n-1)(i-1)+i}}{pnm} C_s
\]

\[
+ \left( \frac{p^n - 1}{p - 1} \right) \sum_{i=1}^{m} \sum_{s=m-i+1}^{m} x_i \frac{p^{(n-2)(i-1)+(m-s)+i}}{pnm} C_s.
\]

Changing the summation order (Fubini’s Theorem), we obtain

\[
\sum_{i=1}^{m} x_i M_i = y_0 \text{Id}_G + \left( \frac{p^n - 1}{p - 1} \right) \sum_{i=1}^{m-1} \sum_{s=1}^{m-s} x_i \frac{p^{(n-1)(i-1)+i}}{pnm} C_s
\]

\[
+ \left( \frac{p^n - 1}{p - 1} \right) \sum_{s=1}^{m} \sum_{i=m-s+1}^{m} x_i \frac{p^{(n-2)(i-1)+(m-s)+i}}{pnm} C_s
\]

\[
= \sum_{s=0}^{m} y_s C_s.
\]
We have, for $1 \leq s \leq m - 1$,

$$y_s = \left(\frac{p^n-1}{p-1}\right) \sum_{i=1}^{m-s} x_i \frac{p^{(n-1)(i-1)+i}}{p^{nm}} + \left(\frac{p^n-1}{p-1}\right) \sum_{i=m-s+1}^{m} x_i \frac{p^{(n-2)(i-1)+(m-s)+i}}{p^{nm}}$$

and

$$y_m = \left(\frac{p^n-1}{p-1}\right) \sum_{i=1}^{m} x_i \frac{p^{(n-2)(i-1)+i}}{p^{nm}}.$$ (3.7)

Consider $1 \leq s \leq m - 2$. Our goal is to show that $x_1, \ldots, x_m$ can be chosen so that $y_s = y_{s+1}$. From (3.6) we obtain

$$x_{m-s} = \frac{p^n \cdot (p^n-1)}{p^{nm}} \sum_{i=m-s+1}^{m} p^{(n-2)(i-1)+i} x_i, \quad 1 \leq s \leq m - 2.$$ (3.8)

Similarly, for $s = m - 1$, we obtain from $y_{m-1} = y_m$ (3.6) and (3.7)

$$x_1 = -(p^n-1) \sum_{i=2}^{m} p^{(n-1)(i-2)} x_i.$$ (3.9)

Taking $s = 1$ in (3.8), we obtain

$$x_{m-1} = -(p^n-1)x_m.$$ (3.10)

From (3.10), taking $s = 2$ in (3.8), we obtain $x_{m-2} = -(p^n-1)x_m$. By induction we obtain

$$x_2 = \cdots = x_{m-1} = -(p^n-1)x_m.$$ (3.11)

Finally, from (3.11) and (3.9) we get $x_1 = -(p^n-1)x_m$.

We let $x_m = 1$ and obtain $x_i = -(p^n-1)$ for $1 \leq i \leq m - 1$. Then, from (3.6) and (3.7) we have

$$y_1 = \cdots = y_m = \left(\frac{p^n-1}{p-1}\right) \frac{1}{p^{nm-1}}.$$
Therefore
\[- \sum_{i=1}^{m-1} \sum_{H \in H_i} (p^{n-1} - 1) \epsilon_H + \sum_{H \in H_m} \epsilon_H = -(p^{n-1} - 1) \sum_{i=1}^{m-1} M_i + M_m\]
\[= y_0 \text{Id}_G + \frac{1}{p^{m-1}} \left( \frac{p^{n-1} - 1}{p - 1} \right) \sum_{s=1}^{m} C_s\]
\[= z_0 \epsilon_{\text{Id}_G} + \frac{1}{p^{m-1}} \left( \frac{p^{n-1} - 1}{p - 1} \right) p^n \epsilon_G\]
\[= z_0 \epsilon_{\text{Id}_G} + p \left( \frac{p^{n-1} - 1}{p - 1} \right) \epsilon_G,\]
(3.12)

where \(z_0 = y_0 - \left( \frac{p^{n-1} - 1}{p - 1} \right) \frac{1}{p^{m-1}}.\) Since \(y_0 = \sum_{i=1}^{m} \frac{p^{i-1}}{p^m} x_i; h_i\) with \(x_i\) as in (3.10) and (3.11) with \(x_m = 1,\) we obtain \(z_0 = 1.\)

Theorem 2.2 is now a consequence of Theorem 2.1 and (3.12).

To prove Theorem 2.4, we consider now \(T_i, 0 \leq i \leq m.\) We have \(|T_i| = h_i.\) Let
\[Q_i := \sum_{H \in T_i} \epsilon_H.\]

Consider an element \(\sigma \in G_s.\) Let \(N(i, \sigma)\) be the number of cyclic subgroups of \(G\) of order \(p^i\) containing \(\sigma.\) Since for any two elements of \(G_s,\) there exists an automorphism of \(G\) sending one into the other, as in (3.2), it makes sense to define
\[N(i, s) := N(i, \sigma),\]
where \(\sigma\) is any element of \(G_s.\)

Then
\[Q_i = \frac{1}{p^i} \sum_{H \in T_i} \sum_{\sigma \in H} \sigma\]
\[= \frac{1}{p^i} \sum_{s=0}^{m} N(i, s) \sum_{\sigma \in G_s} \sigma\]
\[= \frac{1}{p^i} \sum_{s=0}^{m} N(i, s) C_s.\]
(3.13)
First we compute \( N(m, s) \). Let \( \{\tau_1, \ldots, \tau_n\} \) be a basis of \( G \) over \( \mathbb{Z}/p^m\mathbb{Z} \). More precisely, \( G = \langle \tau_1, \ldots, \tau_n \rangle \) and \( o(\tau_j) = p^m \) for \( 1 \leq j \leq n \). Let \( \mu \in G \), say \( \mu = \tau_1^{\alpha_1} \cdots \tau_n^{\alpha_n} \). Then \( o(\mu) = p^m \) if and only if there exists \( 1 \leq j \leq n \) such that \( \gcd(\alpha_j, p) = 1 \). Fix an element \( \sigma \) of \( G_s \) with \( s \geq 1 \). We can choose the basis \( \{\tau_1, \ldots, \tau_n\} \) of \( G \) such that \( \tau_1^{p^m-s} = \sigma \).

We have \( h_m = \frac{p^m-\alpha_{n-1}}{p^m-1} \). The different \( h_m \) cyclic subgroups of \( G \) of order \( p^m \) are

\[
\langle \tau_1^{\alpha_1^{m-1}} \cdots \tau_n^{\alpha_n^{m-1}} \rangle, \quad 0 \leq \alpha_j \leq p^m - 1, \quad 2 \leq j \leq n,
\]

\[
\langle \tau_1^{\alpha_1^{m-1}} \tau_2^{\alpha_2} \cdots \tau_n^{\alpha_n^{m-1}} \rangle, \quad 0 \leq \alpha_1 \leq p^{m-1} - 1 \quad \text{and} \quad 0 \leq \alpha_j \leq p^m - 1, \quad 3 \leq j \leq n,
\]

\[
\vdots
\]

\[
\langle \tau_1^{\alpha_1^{m-1}} \tau_2^{\alpha_2} \cdots \tau_{k-1}^{\alpha_{k-1}} \tau_k^{\alpha_k+1} \cdots \tau_n^{\alpha_n^{m-1}} \rangle, \quad 0 \leq \alpha_j \leq p^{m-1} - 1, \quad 1 \leq j \leq k - 1
\]

\[
\vdots
\]

\[
\langle \tau_1^{\alpha_1^{m-1}} \tau_2^{\alpha_2} \cdots \tau_{n-1}^{\alpha_{n-1}} \rangle, \quad 0 \leq \alpha_j \leq p^{m-1} - 1, \quad 1 \leq j \leq n - 1.
\]

Note that \( \sigma \) does not belong to any subgroup of the form

\[
\langle \tau_1^{\alpha_1^{m-1}} \tau_2^{\alpha_2} \cdots \tau_{k-1}^{\alpha_{k-1}} \tau_k^{\alpha_k+1} \cdots \rangle, \quad k \geq 2,
\]

since \( s \geq 1 \). Otherwise we would have

\[
\sigma = \tau_1^{p^m-\beta} = \langle \tau_1^{p^m-\alpha_1^{m-1}} \tau_2^{\alpha_2} \cdots \rangle^\beta
\]

for some \( 0 \leq \beta \leq p^m - 1 \). Since \( \{\tau_1, \ldots, \tau_n\} \) is a basis of \( G \), we would have that \( p^m \mid \beta \), that is \( \beta = 0 \) which is impossible since \( \sigma \neq \text{Id}_G \).

Similarly, we have \( \sigma \in \langle \tau_1^{\alpha_1^{m-1}} \cdots \rangle \) if and only if \( \alpha_j = p^m \) with \( 0 \leq \alpha_j \leq p^{m-1} - 1, \quad 2 \leq j \leq n \). For \( s = 0 \) we have \( \sigma = \text{Id}_G \) and \( N(m, 0) = h_m \).

Therefore, we have

\[
N(m, s) = \begin{cases} 
  p^{(m-s)(n-1)}, & 1 \leq s \leq m, \\
  h_m, & s = 0.
\end{cases}
\]  

(3.14)

Now let \( 0 \leq i \leq m \). If \( i < s \), then \( |H| = p^i < p^s = o(\sigma) \) so that \( \sigma \notin H \). Thus \( N(i, s) = 0 \) if \( i < s \). Now let \( s \leq i \). If \( s = 0 \) then
\[ N(i, 0) = h_i, \text{ since } \sigma = \text{Id}_G. \] Next, we consider \( s \geq 1 \). Let \( 1 \leq t \leq m \) and \( \phi_t : G \rightarrow G, \phi(x) = x^{p^t} \). Then \( \ker \phi_t = \{ x \in G \mid x^{p^t} = 1 \} = \{ x \in G \mid o(x) \text{ divides } p^t \} \) and the image of \( \phi_t \) is \( G^{p^t} \). In particular if \( t = i \), then any \( H \in T_i \) satisfies \( H \subseteq \ker \phi_i \). It is easy to see that \( \ker \phi_i = G^{p^m} \cong \left( \mathbb{Z}/p^m \mathbb{Z} \right)^n \). Therefore, from the case \( i = m \), we have \( N(i, s) = p^{(i-s)(n-1)} \) for \( s \neq 0 \) and \( N(i, 0) = h_i \). From (3.14) we get

\[
N(i, s) = \begin{cases} 
  h_i, & s = 0, \quad 0 \leq i \leq m, \\
  p^{(i-s)(n-1)}, & 1 \leq s \leq i \leq m, \\
  0, & 0 \leq i < s \leq m.
\end{cases}
\]

(3.15)

From (3.13) and (3.15) we obtain

\[
Q_i = \frac{1}{p^i} \sum_{s=0}^{i} N(i, s)C_s = \frac{1}{p^i} h_i \text{Id}_G + \sum_{s=1}^{i} p^{(i-s)(n-1)-i} C_s.
\]

Equivalently, we have

\[
p^i Q_i = h_i \text{Id}_G + \sum_{s=1}^{i} p^{(i-s)(n-1)} C_s, \quad 0 \leq i \leq m, \quad Q_0 = \text{Id}_G.
\]

(3.16)

Let \( x_1, \ldots, x_n \in \mathbb{Q} \) be such that \( \sum_{i=1}^{m} x_i p^i Q_i = y_0 \text{Id}_G + \sum_{s=1}^{m} y_s C_s \) with \( y_0 \in \mathbb{Q} \) and \( y_1 = y_2 = \cdots = y_m \neq 0 \). Then, from (3.16), we have

\[
\sum_{i=1}^{m} x_i p^i Q_i = \left( \sum_{i=1}^{m} x_i h_i \right) \text{Id}_G + \sum_{s=1}^{m} x_s p^{(i-s)(n-1)} C_s
\]

\[
= y_0 \text{Id}_G + \sum_{s=1}^{m} \sum_{i=s}^{m} x_i p^{(i-s)(n-1)} C_s = y_0 \text{Id}_G + \sum_{s=1}^{m} y_s C_s,
\]

where \( y_0 = \sum_{i=1}^{m} x_i h_i \) and for \( s \geq 1 \),

\[
y_s = \sum_{i=s}^{m} x_i p^{(i-s)(n-1)} = x_s + \sum_{i=s+1}^{m} x_i p^{(i-s)(n-1)}.
\]

From the condition \( y_1 = \cdots = y_m \), we obtain, by induction on \( s \), that

\[
x_1 = x_2 = \cdots = x_{m-1} = -(p^{n-1} - 1)x_m.
\]
We take $x_m = 1$ and get $x_i = -(p^{n-1} - 1)$, $1 \leq i \leq m - 1$. With these values, we obtain $y_1 = y_2 = \cdots = y_m = 1$ and $y_0 = \frac{p^{n-1}}{p-1}$.

Then, we finally obtain a relation among idempotents of $T_i$, $0 \leq i \leq m$:

$$-(p^{n-1} - 1) \sum_{i=1}^{m-1} p^i \epsilon_H + \sum_{H \in T_m} p^m \epsilon_H = \left( \frac{p^{n-1}}{p-1} - 1 \right) \epsilon_{1G} + p^{nm} \epsilon_G$$

$$= p \left( \frac{p^{n-1} - 1}{p-1} \right) \epsilon_{1G} + p^{nm} \epsilon_G.$$

Theorem 2.4 follows from Kani’s Theorem (Theorem 2.1).

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